

Modern Physics Letters A  
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## General Relativity for $N$ Properties.

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Received (Day Month Year)  
 Revised (Day Month Year)

We determine the coefficients of the terms multiplying the gauge fields, gravitational field and cosmological term in a scheme whereby properties are characterized by  $N$  anticommuting scalar Grassmann variables. We do this for general  $N$ , using analytical methods; this obviates the need for our algebraic computing package which can become quite unwieldy as  $N$  is increased.

*Keywords:* Grassmann; property; relativity.

PACS Nos.: include PACS Nos.

### 1. Structure of the Lagrangian terms

Over the last few years we have developed a scheme whereby the gravity is unified with other force fields by embracing them all in a supermetric which features space-time augmented by Lorentz scalar anticommuting coordinates  $\zeta^\mu$ . These  $\zeta$  specify the characteristics or properties of an event,<sup>1–4</sup> in addition to location and time  $x^m$ . This scheme<sup>a</sup> resulted in a Lagrangian which contained the Yang-Mills, gravitational and cosmological terms, consistent with general coordinate invariance, but which pointed to the need for coupling constant unification at an appropriately high length scale  $l$  at least when chirality was involved. We were able to make progress for the case of a small number of properties by using of an algebraic computation package like Mathematica; this provided ample confirmation for the gauge invariance of the result by explicit computation of the super-Ricci scalar. The final answer was dependent on a number of property curvature coefficients and the calculation became progressively more difficult (and expensive in machine time) as the number  $N$  of properties got larger, in a factorial sense. In this letter we shall describe a way of extracting the result for any  $N$  by an analytical method which significantly obviates the need for extensive computer calculation and is a major advance in tackling

<sup>a</sup>One can always make any gauge field Lagrangian consistent with general relativity by ensuring invariance under spacetime coordinate changes through the gravitational metric or vierbein but consistency does not mean proper unification.

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the practical case of  $N = 5$  (or  $N=10$  if we distinguish chiralities) which comprises the standard model.

The trick which leads to such an advance is based upon the Palatini form of the Ricci superscalar  $\mathcal{R}$ :

$$\mathcal{R} = (-1)^{[L]} G^{MK} [(-1)^{[L][M]} \Gamma_{KL}^N \Gamma_{NM}^L - \Gamma_{KM}^N \Gamma_{NL}^L], \quad (1)$$

where  $M = m$  for spacetime or  $M = \mu$  for graded property; the Christoffel symbol  $\Gamma$  is defined by

$$2\Gamma_{MN}^K \equiv [(-1)^{[M][N]} G_{MR,N} + G_{NR,M} - (-1)^{[R]([M]+[N])} G_{MN,R}] (-1)^{[R]} G^{RK}, \quad (2)$$

all derivatives are left-sided and the square brackets denote the grading (0 or 1) of the appropriate index. The significant remark is that  $\mathcal{R}$  formally possesses the structure  $G^{KL} G^{MN} G^{RS} (\partial.G.\partial.G..)_{KLMNRS}$  with the index subscripts  $KLMNRS$  distributed between the derivatives and metrics within the brackets ( ). It then becomes an exercise in picking out typical terms which can contribute to the gravitational curvature, the gauge fields and the cosmological constant.

## 2. The supermetric

As explained in previous work the metric (derived from triangular frame vectors) which contains gravity and the gauge fields (denoted by  $A$  and ignoring coupling constants) has the following components:

$$x - x \text{ sector, } G_{mn} = g_{mn} C + l^2 \bar{\zeta} (A_m A_n + A_n A_m) \zeta C' / 2, \quad (3)$$

$$x - \zeta \text{ sector, } G_{m\nu} = -il^2 (\bar{\zeta} A_m)^{\bar{\nu}} C' / 2, \quad (4)$$

$$\zeta - \bar{\zeta} \text{ sector, } G_{\mu\bar{\nu}} = l^2 \delta_{\mu}^{\nu} C' / 2. \quad (5)$$

In the most general situation, expressions  $C, C'$  represent polynomials of permitted *gauge-invariant* curvature terms:

$$C \equiv 1 + \sum_{r=1}^N c_r Z^r, \quad C' \equiv 1 + \sum_{r=1}^N c'_r Z^r; \quad Z \equiv \bar{\zeta} \zeta. \quad (6)$$

The inverse metric components are readily found:

$$x - x \text{ sector, } G^{mn} = g^{mn} C^{-1}, \quad (7)$$

$$x - \zeta \text{ sector, } G^{m\nu} = i(A^m \zeta)^{\nu} C^{-1}, \quad (8)$$

$$\zeta - \bar{\zeta} \text{ sector, } G^{\mu\bar{\nu}} = [2\delta_{\nu}^{\mu} / l^2 - (\bar{\zeta} A^m)^{\bar{\nu}} (A_m \zeta)^{\mu}] C'^{-1}. \quad (9)$$

A general rotation of the property coordinates,  $\zeta \rightarrow \exp[i\Theta(x)]\zeta$  then just corresponds to a gauge transformation of the force fields, so we anticipate that the Ricci superscalar  $\mathcal{R}$  must turn out to be gauge invariant. Indeed it is, as verified multiple times through Mathematica evaluations.

To further the calculation of the dependence of the Lagrangian on the property curvature coefficients  $c_r$  and  $c'_r$ , we will require the Berezinian<sup>5,6</sup> of the metric. The

latter is actually gauge invariant so we can set  $A \rightarrow 0$  to evaluate it. A simple calculation produces

$$\sqrt{G_{..}} = \sqrt{g_{..}} C^2 (l^2 C' / 2)^{-N}. \quad (10)$$

We will also need the Grassmann integrals,

$$\int (d^N \zeta d^N \bar{\zeta}) Z^N = (-1)^{\langle N \rangle} N!, \quad (11)$$

$$\int (d^N \zeta d^N \bar{\zeta}) Z^{N-1} (\bar{\zeta} H \zeta) = (-1)^{\langle N \rangle} (N-1)! \text{Tr } H, \quad (12)$$

where  $\langle M \rangle$  signifies  $\text{int}[M/2]$  and  $H$  stands for a general  $U(N)$  matrix in property space.

Before continuing, we will find it useful to use alternative parametrizations to (6), since they help to simplify the ensuing analysis; namely write

$$C = \exp[-\sum_{r=1}^N a_r Z^r], \quad C' = \exp[-\sum_{r=1}^N a'_r Z^r]; \quad Z \equiv \bar{\zeta} \zeta. \quad (13)$$

Thus derivatives are easily found,

$$\frac{\partial C}{\partial \zeta} = \sum_{r=1}^N \bar{\zeta} r a_r Z^{r-1} C \equiv \bar{\zeta} D C, \quad \frac{\partial C}{\partial \bar{\zeta}} = -\sum_{r=1}^N r a_r Z^{r-1} \zeta C, \equiv -D \zeta C, \text{ etc.} \quad (14)$$

Of course there is a simple translation table between parametrizations (6) and (13):

$$a_1 = -c_1, \quad a_2 = c_1^2/2 - c_2, \quad a_3 = -c_1^3/3 + c_1 c_2 - c_3, \quad (15)$$

$$a_4 = c_1^4/4 - c_1^2 c_2 + c_2/2 + c_1 c_3 - c_4, \quad (16)$$

$$a_5 = -c_1^5/5 + c_1^3 c_2 - c_1 c_2^2 - c_1^2 c_3 + c_2 c_3 - c_5, \quad \text{etc.} \quad (17)$$

In this  $a$ -parametrization,  $\sqrt{G_{..}} = \sqrt{g_{..}} (2/l^2)^N \exp[\sum_{r=1}^N (N a'_r - 2 a_r) Z^r]$ .

### 3. Determination of the various parts of the Lagrangian

We are after the integral of the superscalar curvature,  $\int (d^N \zeta d^N \bar{\zeta}) \sqrt{G_{..}} \mathcal{R}$ , which will produce three sorts of terms: the purely gravitational bit  $R^{[g]}$ , the gauge contribution proportional to  $\text{Tr } F.F$ , where  $F$  is the generalised curl of the gauge field, and finally the constant, cosmological part. The tactic is to identify relevant bits of each by picking out appropriate pieces of  $G^{KL} G^{MN} G^{RS} (\partial.G.. \partial.G..)_{KLMNRS}$ .

#### 3.1. The gravitational term

The gravitational curvature arises from the structure  $g^{kl} g^{mn} g^{rs} (\partial.g.. \partial.g..)_{klmnrs}$ , which itself comes from  $G^{kl} G^{mn} G^{rs} (\partial.G.. \partial.G..)_{klmnrs}$  and therefore carries the factor  $C^{-1}$  as can be ascertained from eqs. (3) and (7). Including the Berezinian, we see that

$$\sqrt{G_{..}} \mathcal{R} \supset \sqrt{g_{..}} R^{[g]} (2/l^2)^N \exp \left[ \sum_1^N (N a'_r - a_r) Z^r \right]. \quad (18)$$

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Upon  $\zeta$  integration we deduce that

$$\int (d^N \zeta d^N \bar{\zeta}) \sqrt{G_{..}} \mathcal{R} \supset \sqrt{g_{..}} R^{[g]} (-1)^{\langle N \rangle} \left( \frac{2}{l^2} \frac{d}{dZ} \right)^N \exp \left[ \sum_1^N (N a'_r - a_r) Z^r \right] \Big|_{Z=0} \quad (19)$$

that we can later convert into  $c$ -form, if desired.

### 3.2. The gauge field term

The curl  $F_{mn} = A_{n,m} - A_{m,n} + i[A_n, A_m]$ , which we are sure arises in the Lagrangian, can be picked out by focussing on the first derivative of the gauge field and ignoring the other parts as these will come automatically. Now the gauge field occurs in the  $x - \zeta$  component  $G_{m\nu}$  of the metric and is attached to a factor of  $\zeta$  as well as  $C'$ . Therefore we need only examine terms of the type  $G^{km} G^{ln} G^{\sigma\bar{\rho}} (\partial_k G_{l\bar{\rho}} \partial_m G_{n\sigma})$  and these engender a factor  $(l^2 C'/2) C^{-2} (\bar{\zeta} F.F \zeta)$  upon contraction over indices. It follows that, apart from a proportionality factor,

$$\sqrt{G_{..}} \mathcal{R} \supset \sqrt{g_{..}} (2/l^2)^{N-1} (\bar{\zeta} F.F \zeta) \exp \left[ \sum_{r=1}^N (N-1) a'_r Z^r \right]. \quad (20)$$

Integration over property (see eq. (12)) yields

$$\int (d^N \zeta d^N \bar{\zeta}) \sqrt{G_{..}} \mathcal{R} \propto \sqrt{g_{..}} \text{Tr}(F^{mn} F_{mn}) (-1)^{\langle N \rangle} \left( \frac{2}{l^2} \frac{d}{dZ} \right)^{N-1} \exp \left[ \sum_{r=1}^N (N-1) a'_r Z^r \right] \Big|_{Z=0}. \quad (21)$$

One readily checks via the case  $N = 1$  that the proportionality factor needed is just  $-1/2$ .

### 3.3. The cosmological term

This piece, which like  $\sqrt{G}$  does not depend on  $A$ , is a bit more complicated because it can arise from three types of contribution:

$$\begin{aligned} & (G^{kl} G^{mn} \text{ or } G^{km} G^{ln}) G_{kl,\rho} G_{mn,\bar{\sigma}} G^{\bar{\sigma}\rho}, \\ & (G^{kl} G_{kl,\bar{\mu}} G_{\rho\bar{\sigma},\nu}) (G^{\nu\bar{\mu}} G^{\bar{\sigma}\rho} \text{ or } G^{\rho\bar{\mu}} G^{\bar{\sigma}\nu}) \\ & (G^{\kappa\bar{\lambda}} G_{\bar{\lambda}\kappa,\bar{\mu}} G_{\rho\bar{\sigma},\nu}) (G^{\nu\bar{\mu}} G^{\bar{\sigma}\rho} \text{ or } G^{\rho\bar{\mu}} G^{\bar{\sigma}\nu}). \end{aligned}$$

Each of these has to be taken with with multiplicative factors  $\alpha_N, \beta_N, \gamma_N$  respectively and may depend on  $N$  through the contraction  $G^{\bar{\rho}\nu} G_{\nu\bar{\rho}} \propto N$ . Thus we can ascertain via comparison with the  $N = 1, 2$  and  $3$  cases that  $\alpha_N = 6$  is  $N$ -independent,  $\beta_N$  is linear in  $N$ , namely  $\beta_N = -4(2N + 1)$ ; and that  $\gamma_N = (2N + 1)(N + 1)$  is  $N$ -quadratic. (Alternatively  $\alpha, \beta, \gamma$  can be painstakingly determined from first principles.) Now

$$G^{kl} G^{mn} G_{kl,\rho} G_{mn,\bar{\sigma}} G^{\bar{\sigma}\rho} \text{ entrains } (2/l^2) C'^{-1} C^{-2} \left( Z \frac{dC}{dZ} \frac{dC}{dZ} \right),$$

$$G^{kl} G_{kl, \bar{\mu}} G_{\rho \bar{\sigma}, \nu} G^{\nu \bar{\mu}} G^{\bar{\sigma} \rho} \text{ entrains } (2/l^2) C'^{-2} C^{-1} \left( Z \frac{dC}{dZ} \frac{dC'}{dZ} \right),$$

$$G^{\kappa \bar{\lambda}} G_{\bar{\lambda} \kappa, \bar{\mu}} G_{\rho \bar{\sigma}, \nu} \text{ entrains } (2/l^2) C'^{-2} C^{-1} \left( Z \frac{dC'}{dZ} \frac{dC'}{dZ} \right).$$

Altogether we can conclude that the cosmological term arises from

$$\sqrt{G..} \mathcal{R} \supset \sqrt{g..} (2/l^2)^{N+1} Z. \exp \left[ \sum_{r=1}^N ((N+1)a'_r - 2a_r) Z^r \right] [\alpha_N D^2 + \beta_N D D' + \gamma_N D'^2], \quad (22)$$

with

$$\alpha_N = 6, \quad \beta_N = -4(2N+1), \quad \gamma_N = (2N+1)(N+1).$$

All that is left is to integrate (12) over property.

### 3.4. The full result for any $N$

Upon  $\zeta$  integration we end up with the totality,

$$\begin{aligned} (-1)^{\langle N \rangle} \left( \frac{l^2}{2} \right)^N \int (d^N \zeta d^N \bar{\zeta}) \sqrt{G..} \mathcal{R} &= \sqrt{g..} \left( \frac{d}{dZ} \right)^N \left( R^{[g]} e^{\sum (N a'_r - a_r) Z^r} \right. \\ &\quad \left. - \frac{l^2}{4N} \text{Tr} F.F Z e^{\sum (N-1) a'_r Z^r} \right. \\ &\quad \left. + \frac{2Z}{l^2} e^{\sum ((N+1) a'_r - 2a_r) Z^r} \{ 6D^2 - 4(2N+1) D D' + (2N+1)(N+1) D^2 \} \right) \Big|_{Z=0} \end{aligned} \quad (23)$$

Converting from  $a_r$  to  $c_r$  via (15) to (17), the reader can verify that the results for  $N = 1, 2, 3$  stated in previous papers emerge correctly. These are tabled below.

Table 1. Coefficients of terms multiplying gravity, gauge and cosmological pieces (up to  $N = 3, C = C'$ )

$N$	$R^{[g]}$	$\text{Tr} F.F$	cosmological constant
1	$(2/l^2)(c_1 - c'_1)$	$-1/2$	$(24/l^4)(c_1 - c'_1)^2$
2	$(8/l^4)(2c_1 c'_1 - 3c_1'^2 - c_2 + 2c'_2)$	$-c'_1/l^2$	$-(16/l^6)(24c_1 c_2 - 38c_1'^2 c'_1 - 40c_2 c'_1 + 110c_1 c_1'^2 - 75c_1'^3 - 40c_1 c'_2 + 60c'_1 c'_2)$
3	$(96/l^6)(2c_1^3 - 3c_1 c_2 + c_3)$	$(4/l^4)(3c_1^2 - 2c_2)$	$-(1152/l^8)(5c_1^4 - 10c_1^2 + 2c_2^2 + 3c_1 c_3)$

In this way, the coefficients can be fully determined analytically for any  $N$  without resorting to algebraic computer packages.

## 4. Application to $N = 4$

We may apply the above technique to the case where charge and colour (electricity and chromicity properties) are taken together. Since QED and QCD are parity

invariant we need not concern ourselves with different properties for handedness, as one would need to for electroweak theory. Let indices on  $\zeta$  of  $i = 1, 2, 3$  refer to colour (red, green, blue, carrying charge  $1/3$  as per  $\bar{D}$  quarks) and 4 refer to electronic charge (-1). This means combining the coupling of the gluon fields  $B_m$  with the electromagnetic field  $A_m$  in the property-spacetime sector:

$$G_{m4} = il^2 \zeta^{\bar{4}} e A_m / 2; \quad G_{mi} = il^2 [-\zeta^{\bar{i}} e A_m / 3 + \zeta^{\bar{j}} f B_m^{j\bar{i}}] / 2. \quad (24)$$

The burning issue is whether we are forced to assume that the couplings  $e$  and  $f$  must merge at some high energy scale in order to ensure gravitational universality (as we needed to when discussing chirality) or whether there is sufficient freedom allowing them to differ from each other. In fact we shall presently see that the latter holds.

To that end we will *simplify* the argument by adopting an overall set of curvatures  $C = C'$  arising in (3) to (5). However, because we are dealing with a direct product  $U(1) \times SU(3)$  gauge group, we have at our disposal two independent property invariants:  $\zeta^{\bar{4}} \zeta^4$  and  $\zeta^{\bar{i}} \zeta^i$ . Of particular interest is the possibility of

$$C = 1 + \dots + c_e (\zeta^{\bar{4}} \zeta^4) (\zeta^{\bar{i}} \zeta^i)^2 + c_f (\zeta^{\bar{i}} \zeta^i)^3 + \dots,$$

involving two curvature constants  $c_e$  and  $c_f$ . As we are interested in the gauge field contributions to the Lagrangians, we must focus on terms having the structure  $G^{\mu\bar{\nu}} G^{kl} G^{mn} (\partial_m G_{k\bar{\nu}}) (\partial_n G_{l\bar{\nu}})$ , which entrain a overall factor  $C^{-1}$  multiplying the flat field case. In particular we find that

$$\begin{aligned} G^{4\bar{4}} G^{kl} G^{mn} (\partial_m G_{k\bar{4}}) (\partial_n G_{l\bar{4}}) &\rightarrow g^{km} g^{ln} e^2 \zeta^{\bar{4}} F_{kl} F_{mn} \zeta^4 \\ G^{i\bar{j}} G^{kl} G^{mn} (\partial_m G_{k\bar{j}}) (\partial_n G_{l\bar{i}}) &\rightarrow g^{km} g^{ln} [e^2 \zeta^{\bar{4}} F_{kl} F_{mn} \zeta^4 / 3 + f^2 \zeta^{\bar{i}} (E_{kl} E_{mn})^{i\bar{j}} \zeta^j] \end{aligned} \quad (25)$$

where  $F_{mn} \equiv A_{n,m} - A_{m,n}$  and  $E_{mn} \equiv B_{n,m} - B_{m,n} + if[B_n, B_m]$  are the standard ‘‘curls’’ of the electromagnetic and colour fields respectively.

Now remembering that for four properties  $\sqrt{G..} = (2/l^2)^4 \sqrt{g..} C^{-2}$  we deduce that the sum of the gauge field contributions will be held in the expression

$$\begin{aligned} \mathcal{R} \sqrt{G..} &\supset [1 - 3c_e (\zeta^{\bar{4}} \zeta^4) (\zeta^{\bar{i}} \zeta^i)^2 - 3c_f (\zeta^{\bar{i}} \zeta^i)^3 + \dots] \\ &\quad g^{km} g^{ln} [4e^2 \zeta^{\bar{4}} F_{kl} F_{mn} \zeta^4 / 3 + f^2 \zeta^{\bar{i}} (E_{kl} E_{mn})^{i\bar{j}} \zeta^j]. \end{aligned} \quad (26)$$

It only remains to integrate over the four properties to discover the gauge field Lagrangian, (including appropriate factors of  $l^2$ )

$$\int (d^4 \zeta d^4 \bar{\zeta}) \sqrt{G..} \mathcal{R} \supset -(12/l^2) [4c_f e^2 F.F + c_e f^2 \text{Tr}(E.E)]. \quad (27)$$

Clearly all one needs to do is to set  $c_e f^2 = 4c_f e^2$  and we maintain a uniform gravitational constant in the ensuing work, *without forcing equality of the colour and electromagnetic couplings*. Relaxing the assumptions  $C = C'$  and the form of  $C$  makes it even easier to ensure uniformity of  $G_N$ .

## 5. Conclusions

We have described a ‘first-principles’ way of determining the Ricci coefficients for spacetime curvature, property curvature (embodied in the cosmological constant) and gauge field Lagrangians, which arise from the super-Ricci scalar. The method represents a major advance as it unshackles us from relying on a computer algebra package, which struggles timewise as the number of properties rises. The final results just depend on the property curvature coefficients which enter the supermetric while maintaining gauge covariance and they have been listed in Table 1. The procedure puts us in a strong position for handling electroweak theory and the full standard model unification with gravity.

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